

# Discrete local holomorphic dynamics

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## 1. Introduction

Let  $M$  be a complex manifold, and  $p \in M$ . In this survey, a *(discrete) holomorphic local dynamical system* at  $p$  will be a holomorphic map  $f: U \rightarrow M$  such that  $f(p) = p$ , where  $U \subseteq M$  is an open neighbourhood of  $p$ ; we shall also assume that  $f \neq \text{id}_U$ . We shall denote by  $\text{End}(M, p)$  the set of holomorphic local dynamical systems at  $p$ .

**Remark 1.1:** Since we are mainly concerned with the behavior of  $f$  nearby  $p$ , we shall sometimes replace  $f$  by its restriction to some suitable open neighbourhood of  $p$ . It is possible to formalize this fact by using germs of maps and germs of sets at  $p$ , but for our purposes it will be enough to use a somewhat less formal approach.

**Remark 1.2:** In this survey we shall never have the occasion of discussing continuous holomorphic dynamical systems (i.e., holomorphic foliations). So from now on all dynamical systems in this paper will be discrete, except where explicitly noted otherwise.

To talk about the dynamics of an  $f \in \text{End}(M, p)$  we need to define the iterates of  $f$ . If  $f$  is defined on the set  $U$ , then the second iterate  $f^2 = f \circ f$  is defined on  $U \cap f^{-1}(U)$  only, which still is an open neighbourhood of  $p$ . More generally, the  $k$ -th iterate  $f^k = f \circ f^{k-1}$  is defined on  $U \cap f^{-1}(U) \cap \dots \cap f^{-(k-1)}(U)$ . Thus it is natural to introduce the *stable set*  $K_f$  of  $f$  by setting

$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

Clearly,  $p \in K_f$ , and so the stable set is never empty (but it can happen that  $K_f = \{p\}$ ; see the next section for an example). The stable set of  $f$  is the set of all points  $z \in U$  such that the *orbit*  $\{f^k(z) \mid k \in \mathbb{N}\}$  is well-defined. If  $z \in U \setminus K_f$ , we shall say that  $z$  (or its orbit) *escapes* from  $U$ .

Thus the first natural question in local holomorphic dynamics is:

(Q1) *What is the topological structure of  $K_f$ ?*

For instance, when does  $K_f$  have non-empty interior? As we shall see in section 4, holomorphic local dynamical systems such that  $p$  belongs to the interior of the stable set enjoy special properties; we shall then say that  $p$  is *stable* for  $f \in \text{End}(M, p)$  if it belongs to the interior of  $K_f$ .

**Remark 1.3:** Both the definition of stable set and Question 1 (as well as several other definitions or questions we shall meet later on) are topological in character; we might state them for local dynamical systems which are continuous only. As we shall see, however, the *answers* will strongly depend on the holomorphicity of the dynamical system.

Clearly, the stable set  $K_f$  is *completely  $f$ -invariant*, that is  $f^{-1}(K_f) = K_f$  (this implies, in particular, that  $f(K_f) \subseteq K_f$ ). Therefore the pair  $(K_f, f)$  is a discrete dynamical system in the usual sense, and so the second natural question in local holomorphic dynamics is

(Q2) *What is the dynamical structure of  $(K_f, f)$ ?*

For instance, what is the asymptotic behavior of the orbits? Do they converge to  $p$ , or have they a chaotic behavior? Is there a dense orbit? Do there exist proper  *$f$ -invariant* subsets, that is sets  $L \subset K_f$  such that  $f(L) \subseteq L$ ? If they do exist, what is the dynamics on them?

To answer all these questions, the most efficient way is to replace  $f$  by a “dynamically equivalent” but simpler (e.g., linear) map  $g$ . In our context, “dynamically equivalent” means “locally conjugated”; and we have at least three kinds of conjugacy to consider.

Let  $f_1: U_1 \rightarrow M_1$  and  $f_2: U_2 \rightarrow M_2$  be two holomorphic local dynamical systems at  $p_1 \in M_1$  and  $p_2 \in M_2$  respectively. We shall say that  $f_1$  and  $f_2$  are *holomorphically* (respectively, *topologically*) *locally conjugated* if there are open neighbourhoods  $W_1 \subseteq U_1$  of  $p_1$ ,  $W_2 \subseteq U_2$  of  $p_2$ , and a biholomorphism (respectively, a homeomorphism)  $\varphi: W_1 \rightarrow W_2$  with  $\varphi(p_1) = p_2$  such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi \quad \text{on} \quad \varphi^{-1}(W_2 \cap f_2^{-1}(W_2)) = W_1 \cap f_1^{-1}(W_1).$$

In particular we have

$$\forall k \in \mathbb{N} \quad f_1^k = \varphi^{-1} \circ f_2^k \circ \varphi \quad \text{on} \quad \varphi^{-1}(W_2 \cap \dots \cap f_2^{-(k-1)}(W_2)) = W_1 \cap \dots \cap f_1^{-(k-1)}(W_1),$$

and thus  $K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}})$ . So the local dynamics of  $f_1$  about  $p_1$  is to all purposes equivalent to the local dynamics of  $f_2$  about  $p_2$ .

**Remark 1.4:** Using local coordinates centered at  $p \in M$  it is easy to show that any holomorphic local dynamical system at  $p$  is holomorphically locally conjugated to a holomorphic local dynamical system at  $O \in \mathbb{C}^n$ , where  $n = \dim M$ .

Whenever we have an equivalence relation in a class of objects, there are obvious classification problems. So the third natural question in local holomorphic dynamics is

- (Q3) *Find a (possibly small) class  $\mathcal{F}$  of holomorphic local dynamical systems at  $O \in \mathbb{C}^n$  such that every holomorphic local dynamical system  $f$  at a point in an  $n$ -dimensional complex manifold is holomorphically (respectively, topologically) locally conjugated to a (possibly) unique element of  $\mathcal{F}$ , called the holomorphic (respectively, topological) normal form of  $f$ .*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family  $\mathcal{F}$  of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

- (Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects to any holomorphic local dynamical system  $f$  at  $O \in \mathbb{C}^n$ , called the invariants of  $f$ , so that two holomorphic local dynamical systems at  $O$  can be holomorphically conjugated only if they have the same invariants. The class of invariants is furthermore said complete if two holomorphic local dynamical systems at  $O$  are holomorphically conjugated if and only if they have the same invariants.*

As remarked before, up to now all the questions we asked make sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

A holomorphic local dynamical system at  $O \in \mathbb{C}^n$  is clearly given by an element of  $\mathbb{C}_0\{z_1, \dots, z_n\}^n$ , the space of  $n$ -uples of converging power series in  $z_1, \dots, z_n$  without constant terms. The space  $\mathbb{C}_0\{z_1, \dots, z_n\}^n$  is a subspace of the space  $\mathbb{C}_0[[z_1, \dots, z_n]]^n$  of  $n$ -uples of formal power series without constant terms. An element  $\Phi \in \mathbb{C}_0[[z_1, \dots, z_n]]^n$  has an inverse (with respect to composition) still belonging to  $\mathbb{C}_0[[z_1, \dots, z_n]]^n$  if and only if its linear part is a linear automorphism of  $\mathbb{C}^n$ . We shall say that two holomorphic local dynamical systems  $f_1, f_2 \in \mathbb{C}_0\{z_1, \dots, z_n\}^n$  are *formally conjugated* if there exists an invertible  $\Phi \in \mathbb{C}_0[[z_1, \dots, z_n]]^n$  such that  $f_1 = \Phi^{-1} \circ f_2 \circ \Phi$  in  $\mathbb{C}_0[[z_1, \dots, z_n]]^n$ .

It is clear that two holomorphically locally conjugated holomorphic local dynamical systems are both formally and topologically locally conjugated too. On the other hand, we shall see examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is

- (Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at  $O \in \mathbb{C}^n$ .*

In this survey we shall present some of the main results known on these questions, starting from the one-dimensional situation. But before entering the main core of this paper I would like to heartily thank Mohamad Pouryayevali for the wonderful and very warm hospitality I had the pleasure to enjoy during my stay in Iran.

## 2. One complex variable: the hyperbolic case

Let us then start by discussing holomorphic local dynamical systems at  $0 \in \mathbb{C}$ . As remarked in the previous section, such a system is given by a converging power series  $f$  without constant term:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\}.$$

The number  $a_1 = f'(0)$  is the *multiplier* of  $f$ .

Since  $a_1 z$  is the best linear approximation of  $f$ , it is sensible to expect that the local dynamics of  $f$  will be strongly influenced by the value of  $a_1$ . For this reason we introduce the following definitions:

- if  $|a_1| < 1$  we say that the fixed point 0 is *attracting*;
- if  $a_1 = 0$  we say that the fixed point 0 is *superattracting*;
- if  $|a_1| > 1$  we say that the fixed point 0 is *repelling*;
- if  $|a_1| \neq 0, 1$  we say that the fixed point 0 is *hyperbolic*;
- if  $a_1 \in S^1$  is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if  $a_1 \in S^1$  is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

As we shall see in a minute, the dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case. Notice that if 0 is an attracting (we shall discuss the superattracting case momentarily) fixed point for  $f \in \text{End}(\mathbb{C}, 0)$ , then it is a repelling fixed point for the inverse map  $f^{-1} \in \text{End}(\mathbb{C}, 0)$ .

Assume first that 0 is attracting for the holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}, 0)$ . Then we can write  $f(z) = a_1 z + O(z^2)$ , with  $0 < |a_1| < 1$ ; hence we can find a large constant  $C > 0$ , a small constant  $\varepsilon > 0$  and  $0 < \delta < 1$  such that if  $|z| < \varepsilon$  then

$$|f(z)| \leq (|a_1| + C\varepsilon)|z| \leq \delta|z|. \quad (2.1)$$

In particular, if  $\Delta_\varepsilon$  denotes the disk of center 0 and radius  $\varepsilon$ , we have  $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$  for  $\varepsilon > 0$  small enough, and the stable set of  $f|_{\Delta_\varepsilon}$  is  $\Delta_\varepsilon$  itself (in particular, an one-dimensional attracting fixed point is always stable). Furthermore,

$$|f^k(z)| \leq \delta^k |z| \rightarrow 0$$

as  $k \rightarrow +\infty$ , and thus every orbit starting in  $\Delta_\varepsilon$  is attracted by the origin, which is the reason of the name “attracting” for such a fixed point.

If instead 0 is a repelling fixed point, a similar argument (or the observation that 0 is attracting for  $f^{-1}$ ) shows that for  $\varepsilon > 0$  small enough the stable set of  $f|_{\Delta_\varepsilon}$  reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as shown in the following result, which marked the beginning of the theory of holomorphic dynamical systems:

**Theorem 2.1:** (Koenigs, 1884 [Kœ]) *Let  $f \in \text{End}(\mathbb{C}, 0)$  be an one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let  $a_1 \in \mathbb{C}^*$  be its multiplier. Then:*

- (i)  *$f$  is holomorphically (and hence formally) locally conjugated to its linear part  $g(z) = a_1 z$ .*
- (ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*
- (iii)  *$f$  is topologically locally conjugated to the map  $g_<(z) = z/2$  if  $|a_1| < 1$ , and to the map  $g_>(z) = 2z$  if  $|a_1| > 1$ .*

*Sketch of proof:* Let us assume  $0 < |a_1| < 1$ ; if  $|a_1| > 1$  it will suffice to apply the same argument to  $f^{-1}$ .

Put  $\varphi_k = f^k/a_1^k$ ; using (2.1) it is not difficult to show that the sequence  $\{\varphi_k\}$  converges to a holomorphic map  $\varphi: \Delta_\varepsilon \rightarrow \mathbb{C}$  for  $\varepsilon > 0$  small enough. Since  $\varphi'_k(0) = 1$  for all  $k \in \mathbb{N}$ , we have  $\varphi'(0) = 1$  and so, up to possibly shrink  $\varepsilon$ , we can assume that  $\varphi$  is a biholomorphism with its image. Moreover, we have

$$\varphi(f(z)) = \lim_{k \rightarrow +\infty} \frac{f^k(f(z))}{a_1^k} = a_1 \lim_{k \rightarrow +\infty} \frac{f^{k+1}(z)}{a_1^{k+1}} = a_1 \varphi(z),$$

that is  $f = \varphi^{-1} \circ g \circ \varphi$ , as claimed.

Since  $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$  implies  $f'_1(0) = f'_2(0)$ , the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

Finally, if  $|a_1| < 1$  it is easy to build a topological conjugacy between  $g$  and  $g_<$  on  $\Delta_\varepsilon$ : it suffices to choose any homeomorphism  $\varphi$  between the annulus  $\{\varepsilon/2 \leq |z| < \varepsilon\}$  and the annulus  $\{|a_1|\varepsilon \leq |z| < \varepsilon\}$ , and to extend it by induction to a homeomorphism between the annuli  $\{\varepsilon/2^k \leq |z| \leq \varepsilon/2^{k-1}\}$  and  $\{|a_1|^k \varepsilon \leq |z| \leq |a_1|^{k-1} \varepsilon\}$  by requiring

$$\varphi(\tfrac{1}{2}z) = a_1 \varphi(z).$$

Putting finally  $\varphi(0) = 0$  we then get the topological conjugacy we were looking for.  $\square$

Notice that  $g_<(z) = \frac{1}{2}z$  and  $g_>(z) = 2z$  cannot be topologically conjugated, because (for instance) the origin is stable for  $g_<$  and it is not stable for  $g_>$ .

Thus the dynamics in the one-dimensional hyperbolic case is completely clear. The superattracting case can be treated similarly. If 0 is a superattracting point for an  $f \in \text{End}(\mathbb{C}, 0)$ , we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \dots$$

with  $a_r \neq 0$ ; the number  $r \geq 2$  is the *order* of the superattracting point. An argument similar to the one described above shows that for  $\varepsilon > 0$  small enough the stable set of  $f|_{\Delta_\varepsilon}$  still is all of  $\Delta_\varepsilon$ , and the orbits converge (faster than in the attracting case) to the origin. Furthermore, replacing the maps  $\varphi_k$  in the proof of Theorem 2.1 by maps of the form

$$\varphi_k(z) = [f^k(z)]^{1/r^k},$$

for a suitable choice of the  $r^k$ -th root, one can prove the following

**Theorem 2.2:** (Böttcher, 1904 [B]) *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let  $r \geq 2$  be its order. Then:*

- (i)  *$f$  is holomorphically (and hence formally) locally conjugated to the map  $g(z) = z^r$ .*
- (ii) *two such holomorphic local dynamical systems are holomorphically (or topologically) conjugated if and only if they have the same order.*

Therefore the one-dimensional local dynamics about a hyperbolic or superattracting fixed point is completely clear; let us now discuss what happens about a parabolic fixed point.

### 3. One complex variable: the parabolic case

Let  $f \in \text{End}(\mathbb{C}, 0)$  be a (non-linear) holomorphic local dynamical system with a parabolic fixed point at the origin. Then we can write

$$f(z) = e^{2i\pi p/q} z + a_{r+1} z^{r+1} + a_{r+2} z^{r+2} + \dots, \quad (3.1)$$

with  $a_{r+1} \neq 0$ , where  $p/q \in \mathbb{Q} \cap [0, 1)$  is the *rotation number* of  $f$ , and the number  $r+1 \geq 2$  is the *multiplicity* of  $f$  at the fixed point.

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order. Indeed, if we had  $\varphi^{-1} \circ f \circ \varphi(z) = e^{2i\pi p/q} z$  we would have  $\varphi^{-1} \circ f^q \circ \varphi = \text{id}$ , that is  $f^q = \text{id}$ .

In particular, if the rotation number is 0 (that is the multiplier is 1, and we shall say that  $f$  is *tangent to the identity*), then  $f$  *cannot* be locally conjugated to the identity (unless it was the identity to begin with, which is not a very interesting case dynamically speaking). More precisely, the stable set of such an  $f$  is never a neighbourhood of the origin. To understand why, let us first consider a map of the form

$$f(z) = z(1 + az^r)$$

for some  $a \neq 0$ . Let  $v \in S^1 \subset \mathbb{C}$  be such that  $av^r$  is real and positive. Then for any  $c > 0$  we have

$$f(cv) = c(1 + c^r av^r)v \in \mathbb{R}^+ v;$$

moreover,  $|f(cv)| > |cv|$ . In other words, the half-line  $\mathbb{R}^+v$  is  $f$ -invariant and repelled from the origin, that is  $K_f \cap \mathbb{R}^+v = \emptyset$ . Conversely, if  $av^r$  is real and negative then the segment  $[0, |a|^{-1/r}]v$  is  $f$ -invariant and attracted by the origin. So  $K_f$  neither is a neighbourhood of the origin nor reduces to  $\{0\}$ .

This example suggests the following definition. Let  $f \in \text{End}(\mathbb{C}, 0)$  be of the form (3.1) and tangent to the identity. Then a unit vector  $v \in S^1$  is an *attracting* (respectively, *repelling*) *direction* for  $f$  at the origin if  $a_{r+1}v^r$  is real and negative (respectively, positive). Clearly, there are  $r$  equally spaced attracting directions, separated by  $r$  equally spaced repelling directions; furthermore, a repelling (attracting) direction for  $f$  is attracting (repelling) for  $f^{-1}$ , which is defined in a neighbourhood of the origin.

It turns out that to every attracting direction is associated a connected component of  $K_f \setminus \{0\}$ . Let  $v \in S^1$  be an attracting direction for an  $f$  tangent to the identity. The *basin* centered at  $v$  is the set of points  $z \in K_f \setminus \{0\}$  such that  $f^k(z) \rightarrow 0$  and  $f^k(z)/|f^k(z)| \rightarrow v$  (notice that, up to shrinking the domain of  $f$ , we can assume that  $f(z) \neq 0$  for all  $z \in K_f \setminus \{0\}$ ). If  $z$  belongs to the basin centered at  $v$ , we shall say that the orbit of  $z$  *tends to 0 tangent to v*.

A slightly more specialized (but more useful) object is the following: an *attracting petal* centered at an attracting direction  $v$  is an open simply connected  $f$ -invariant set  $P \subseteq K_f \setminus \{0\}$  such that a point  $z \in K_f \setminus \{0\}$  belongs to the basin centered at  $v$  if and only if its orbit intersects  $P$ . In other words, the orbit of a point tends to 0 tangent to  $v$  if and only if it is eventually contained in  $P$ . A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of  $f$ .

It turns out that the basins centered at the attracting directions are exactly the connected components of  $K_f \setminus \{0\}$ , as shown in the *Leau-Fatou flower theorem*:

**Theorem 3.1:** (Leau, 1897 [L]; Fatou, 1919-20 [F1-3]) *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a holomorphic local dynamical system tangent to the identity with multiplicity  $r + 1 \geq 2$  at the fixed point. Let  $v_1, v_3, \dots, v_{2r-1} \in S^1$  be the  $r$  attracting directions of  $f$  at the origin, and  $v_2, v_4, \dots, v_{2r} \in S^1$  the  $r$  repelling directions. Then*

- (i) *There exists for each attracting (repelling) direction  $v_{2j-1}$  ( $v_{2j}$ ) an attracting (repelling) petal  $P_{2j-1}$  ( $P_{2j}$ ), so that the union of these  $2r$  petals together with the origin forms a neighbourhood of the origin. Furthermore, the  $2r$  petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is  $\pi/r$ .*
- (ii)  *$K_f \setminus \{0\}$  is the (disjoint) union of the basins centered at the  $r$  attracting directions.*
- (iii) *If  $P$  is an attracting petal, then  $f|_P$  is holomorphically conjugated to the translation  $z \mapsto z + 1$  defined on a subset of the complex plane containing some right half-plane.*

*Sketch of proof:* Up to a linear change of variables, we can assume that  $a_{r+1} = -1$ , so that the attracting directions are the  $r$ -th roots of unity. For any  $\delta > 0$ , the set  $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$  has exactly  $r$  connected components, each one centered on a different  $r$ -th root of unity; it will turns out that, for  $\delta$  small enough, these connected components are the attracting petals of  $f$ .

Let  $P_\delta$  denote one of these connected components, and let  $\psi: P_\delta \rightarrow \mathbb{C}$  be given by

$$\psi(z) = \frac{1}{rz^r}.$$

This is a biholomorphism of  $P_\delta$  with a right half-plane  $H_\delta = \{w \in \mathbb{C} \mid \text{Re } w > 1/(2r\delta)\}$ , and we have

$$\psi \circ f \circ \psi^{-1}(w) = w + 1 + O(w^{-1/r}). \quad (3.2)$$

Then, setting  $F = \psi \circ f \circ \psi^{-1}$ , it is not difficult to prove that for  $\delta > 0$  small enough the right half-plane  $H_\delta$  is  $F$ -invariant, and that for any  $w \in H_\delta$  the orbit  $\{F^k(w)\}$  converges to  $\infty$  tangent to  $+1$ . Thus it follows that  $P_\delta$  is  $f$ -invariant, and that the orbits in  $P_\delta$  tends to the origin tangent to the central direction  $v$  of  $P_\delta$ . Since every orbit converging to the origin tangent to  $v$  must eventually intersect  $P_\delta$ , every such  $P_\delta$  is an attracting petal.

Arguing in the same way with  $f^{-1}$  we get the repelling petals, and thus (i) follows. Since it is not difficult to prove that every orbit converging to the origin must be tangent to an attracting direction, (ii) follows too. Finally, a subtler argument shows that we can modify  $\psi$  in each petal so to get rid of the term  $O(w^{-1/r})$  in (3.2), proving (iii).  $\square$

So we have a complete description of the dynamics in the neighbourhood of the origin. Actually, Camacho has pushed this argument even further, obtaining a complete topological classification of one-dimensional holomorphic local dynamical systems tangent to the identity:

**Theorem 3.2:** (Camacho, 1978 [C]; Shcherbakov, 1982 [S]) *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a holomorphic local dynamical system tangent to the identity with multiplicity  $r + 1$  at the fixed point. Then  $f$  is topologically locally conjugated to the map*

$$z \mapsto z + z^{r+1}.$$

The formal classification is simple too, though different, and it can be obtained with an easy computation (see, e.g., Milnor [Mi]):

**Proposition 3.3:** *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a holomorphic local dynamical system tangent to the identity with multiplicity  $r + 1$  at the fixed point. Then  $f$  is formally conjugated to the map*

$$z \mapsto z + z^{r+1} + \beta z^{2r+1},$$

where  $\beta$  is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - f(z)}, \quad (3.3)$$

where the integral is taken over a small positive loop  $\gamma$  about the origin.

The number  $\beta$  given by (3.3) is called *index* of  $f$  at the fixed point.

The holomorphic classification is much more complicated: as shown by Voronin [V] and Écalé [É1–2] in 1981, it depends on functional invariants. We shall now try to roughly describe it; see [I2] (and the original papers; see also [K]) for details. Let  $f \in \text{End}(\mathbb{C}, 0)$  be tangent to the identity with multiplicity  $r + 1$  at the fixed point; up to a linear change of coordinates we can assume that  $a_{r+1} = 1$ . Let  $P_1, \dots, P_{2r}$  be a set of petals as in Theorem 3.1.(i), chosen so that  $P_{2r}$  is centered on the positive real semiaxis, and the others are arranged cyclically counterclockwise. Denote by  $H_j$  the biholomorphism conjugating  $f|_{P_j}$  to the shift  $z \mapsto z + 1$  in either a right (if  $j$  is odd) or left (if  $j$  is even) half-plane given by Theorem 3.1.(iii) — applied to  $f^{-1}$  for the repelling petals. If we moreover require that

$$H_j(z) = -\frac{1}{rz^r} + \beta \log z + o(1), \quad (3.4)$$

where  $\beta$  is the index of  $f$  at the origin, then  $H_j$  is uniquely determined. Thus in the sets  $H_j(P_j \cap P_{j+1})$  we can consider the composition  $\tilde{\Phi}_j = H_{j+1} \circ H_j^{-1}$ . It is easy to check that  $\tilde{\Phi}_j(w+1) = \tilde{\Phi}_j(w) + 1$  for  $j = 1, \dots, 2r-1$ , and thus  $\psi_j = \tilde{\Phi}_j - \text{id}$  is a 1-periodic holomorphic function (for  $j = 2r$  we need to take  $\psi_{2r} = \Phi_{2r} = \text{id} + 2\pi i \beta$  to get a 1-periodic function). Hence each  $\psi_j$  can be extended to a suitable upper (if  $j$  is odd) or lower (if  $j$  is even) half-plane. Furthermore, it is possible to prove that the functions  $\psi_1, \dots, \psi_{2r}$  are exponentially decreasing, that is they are bounded by  $\exp(-c|w|)$  as  $|\text{Im } w| \rightarrow +\infty$ , for a suitable  $c > 0$  depending on  $f$ .

Now, if we replace  $f$  by a holomorphic local conjugate  $g = h^{-1} \circ f \circ h$ , and denote by  $G_j$  the corresponding biholomorphisms, it turns out that  $H_j \circ G_j^{-1} = \text{id} + a$  for a suitable  $a \in \mathbb{C}$  independent of  $j$ . This suggests the introduction of an equivalence relation on the set of  $2r$ -uple of functions of the kind  $(\psi_1, \dots, \psi_{2r})$ .

Let  $M_r$  denote the set of  $2r$ -uple of holomorphic 1-periodic functions  $\psi = (\psi_1, \dots, \psi_{2r})$ , with  $\psi_j$  defined in a suitable upper (if  $j$  is odd) or lower (if  $j$  is even) half-plane, and exponentially decreasing when  $|\text{Im } w| \rightarrow +\infty$ . We shall say that  $\psi, \tilde{\psi} \in M_r$  are *equivalent* if there is  $a \in \mathbb{C}$  such that  $\tilde{\psi}_j = \psi_j \circ (\text{id} + a)$  for  $j = 1, \dots, 2r$ . We denote by  $\mathcal{M}_r$  the set of all equivalence classes.

The procedure described above allows us to associate to any  $f \in \text{End}(\mathbb{C}, 0)$  tangent to the identity with multiplicity  $r + 1$  at the fixed point an element  $\mu_f \in \mathcal{M}_r$ , called the *sectorial invariant*. Then the holomorphic classification proved by Écalé and Voronin is

**Theorem 3.4:** (Écalé, 1981 [É1–2]; Voronin, 1981 [V]) *Let  $f, g \in \text{End}(\mathbb{C}, 0)$  be two holomorphic local dynamical systems tangent to the identity. Then  $f$  and  $g$  are holomorphically locally conjugated if and only if they have the same multiplicity, the same index and the same sectorial invariant. Furthermore, for any  $r \geq 1$ ,  $\beta \in \mathbb{C}$  and  $\mu \in \mathcal{M}_r$  there exists  $f \in \text{End}(\mathbb{C}, 0)$  tangent to the identity with multiplicity  $r + 1$ , index  $\beta$  and sectorial invariant  $\mu$ .*

For a sketch of the proof, together with a more geometrical description of the sectorial invariant, see [I2] and [M1–2].

**Remark 3.1:** In particular, holomorphic local dynamical systems tangent to the identity give examples of local dynamical systems that are topologically conjugated without being neither holomorphically nor formally conjugated, and of local dynamical systems that are formally conjugated without being holomorphically conjugated.

Finally, if  $f \in \text{End}(\mathbb{C}, 0)$  satisfies  $a_1 = e^{2\pi i p/q}$ , then  $f^q$  is tangent to the identity. Therefore we can apply the previous results to  $f^q$  and then infer informations about the dynamics of the original  $f$ . See [Mi], [C], [É1–2] and [V] for details.

#### 4. One complex variable: the elliptic case

We are left with the elliptic case:

$$f(z) = e^{2\pi i \theta} z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}, \quad (4.1)$$

with  $\theta \notin \mathbb{Q}$ . It turns out that the local dynamics depends mostly on the numerical properties of  $\theta$ . More precisely, for a full measure subset  $B$  of  $\theta \in [0, 1] \setminus \mathbb{Q}$  all holomorphic local dynamical systems of the form (4.1) are *holomorphically linearizable*, that is holomorphically locally conjugated to their (common) linear part, the irrational rotation  $z \mapsto e^{2\pi i \theta} z$ . Conversely, the complement  $[0, 1] \setminus B$  is a  $G_\delta$ -dense set, and for all  $\theta \in [0, 1] \setminus B$  the quadratic polynomial  $z \mapsto z^2 + e^{2\pi i \theta} z$  is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Bryuno and Yoccoz we are going to describe in this section.

The first worthwhile observation in this setting is that it is possible to give a topological characterization of the holomorphically linearizable local dynamical systems:

**Proposition 4.1:** *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a holomorphic local dynamical system with multiplier  $0 < |\lambda| \leq 1$ . Then  $f$  is holomorphically linearizable if and only if it is topologically linearizable if and only if 0 is stable for  $f$ .*

*Sketch of proof:* Assume that 0 is stable. If  $0 < |\lambda| < 1$ , we already saw that  $f$  is linearizable. If  $|\lambda| = 1$ , set

$$\varphi_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \frac{f^j(z)}{\lambda^j},$$

so that

$$\varphi_k \circ f = \lambda \varphi_{k+1} + \frac{\lambda}{k} (\varphi_{k+1} - f). \quad (4.2)$$

The stability of 0 implies that  $\{\varphi_k\}$  is a normal family in a neighbourhood of the origin, and (4.2) implies that a converging subsequence converges to a conjugation between  $f$  and the rotation  $z \mapsto \lambda z$ .  $\square$

The second important observation is that two elliptic holomorphic local dynamical systems with the same multiplier are always formally conjugated:

**Proposition 4.2:** *Let  $f \in \text{End}(\mathbb{C}, 0)$  be a holomorphic local dynamical system of multiplier  $\lambda = e^{2\pi i \theta} \in S^1$  with  $\theta \notin \mathbb{Q}$ . Then  $f$  is formally conjugated to its linear part.*

*Sketch of proof:* It is an easy computation to prove that there is a unique formal power series

$$h(z) = z + h_2 z^2 + \cdots \in \mathbb{C}[[z]]$$

such that  $h(\lambda z) = f(h(z))$ . For later use we explicitly remark that the coefficients of the formal linearization satisfy

$$h_j = \frac{a_j + X_j}{\lambda^j - \lambda}, \quad (4.3)$$

where  $X_j$  is a polynomial expression in  $a_2, \dots, a_{j-1}, h_2, \dots, h_{j-1}$ .  $\square$

The formal power series linearizing  $f$  is not converging if its coefficients grow too fast. Thus (4.3) links the radius of convergence of  $h$  to the behavior of  $\lambda^j - \lambda$ : if the latter becomes too small, the series defining  $h$  does not converge. This is known as the *small denominators problem* in this context.

It is then natural to introduce the following quantity:

$$\Omega_\lambda(m) = \min_{1 \leq k \leq m} |\lambda^k - 1|,$$

for  $\lambda \in S^1$  and  $m \geq 1$ . Clearly,  $\lambda$  is a root of unity if and only if  $\Omega_\lambda(m) = 0$  for all  $m$  greater or equal to some  $m_0 \geq 1$ ; furthermore,

$$\lim_{m \rightarrow +\infty} \Omega_\lambda(m) = 0$$

for all  $\lambda \in S^1$ .

The first one to actually prove that there are elliptic holomorphic local dynamical systems not linearizable has been Cremer, in 1927 [Cr1]. Later he proved the following:

**Theorem 4.3:** (Cremer, 1938 [Cr2]) *Let  $\lambda \in S^1$  be such that*

$$\limsup_{m \rightarrow +\infty} \left( -\frac{1}{m} \log \Omega_\lambda(m) \right) = +\infty. \quad (4.4)$$

*Then there exists  $f \in \text{End}(\mathbb{C}, 0)$  with multiplier  $\lambda$  which is not holomorphically linearizable. Furthermore, the set of  $\lambda \in S^1$  satisfying (4.4) contains a  $G_\delta$ -dense set.*

*Sketch of proof:* Choose inductively  $a_j \in \{0, 1\}$  so that  $|a_j + X_j| \geq 1/2$  for all  $j \geq 2$ , where  $X_j$  is as in (4.3). Then

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\}$$

while (4.4) implies that the radius of convergence of the formal linearization  $h$  is 0, and thus  $f$  cannot be holomorphically linearizable, as required.

Finally, let  $S(q_0) \subset S^1$  denote the set of  $\lambda = e^{2\pi i \theta} \in S^1$  such that

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{2q!}$$

for some  $p/q \in \mathbb{Q}$  in lowest terms with  $q \geq q_0$ . Then it is not difficult to check that each  $S(q_0)$  is a dense open set in  $S^1$ , and that all  $\lambda \in \bigcap_{q_0 \geq 1} S(q_0)$  satisfy (4.4).  $\square$

On the other hand, Siegel, using the technique of majorant series, in 1942 gave a condition on the multiplier ensuring holomorphic linearizability:

**Theorem 4.4:** (Siegel, 1942 [Si]) *Let  $\lambda \in S^1$  be such that there exists  $\beta \geq 1$  and  $\gamma > 0$  such that*

$$\forall m \geq 2 \quad \frac{1}{\Omega_\lambda(m)} \leq \gamma m^\beta. \quad (4.5)$$

*Then all  $f \in \text{End}(\mathbb{C}, 0)$  with multiplier  $\lambda$  are holomorphically linearizable. Furthermore, the set of  $\lambda \in S^1$  satisfying (4.5) for some  $\beta \geq 1$  and  $\gamma > 0$  is of full Lebesgue measure in  $S^1$ .*

**Remark 4.1:** It is interesting to notice that for generic (in a topological sense)  $\lambda \in S^1$  there is a non-linearizable holomorphic local dynamical system with multiplier  $\lambda$ , while for almost all (in a measure-theoretic sense)  $\lambda \in S^1$  every holomorphic local dynamical system with multiplier  $\lambda$  is holomorphically linearizable.

A bit of terminology is now useful: if  $f \in \text{End}(\mathbb{C}, 0)$  is elliptic, we shall say that the origin is a *Siegel point* if  $f$  is holomorphically linearizable; otherwise it is a *Cremer point*.

Theorem 4.4 suggests the existence of a number-theoretical condition on  $\lambda$  ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier  $\lambda$ . And indeed this is the content of the celebrated *Bryuno-Yoccoz theorem*:



**Theorem 4.5:** *Let  $\lambda \in S^1$ .*

(i) (Bryuno, 1965 [Bry1–3]) *If  $\lambda$  satisfies*

$$\sum_{k=0}^{+\infty} (-2^{-k} \log \Omega_\lambda(2^{k+1})) < +\infty, \quad (4.6)$$

*then the origin is a Siegel point for all  $f \in \text{End}(\mathbb{C}, 0)$  with multiplier  $\lambda$ .*

(ii) (Yoccoz, 1988 [Y1–2]) *If  $\lambda$  does not satisfy (4.6), then the origin is a Cremer point for some  $f \in \text{End}(\mathbb{C}, 0)$  with multiplier  $\lambda$ . In particular, the origin is a Cremer point for  $f(z) = \lambda z + z^2$ .*

The original proof by Bryuno of Theorem 4.5.(i) uses majorant series; see, e.g., [He] and references therein. Yoccoz found a more geometric approach, based on conformal and quasi-conformal geometry, and proved Theorem 4.5.(ii). Furthermore, he showed that the origin is a Siegel point for all elliptic holomorphic local dynamical systems with multiplier  $\lambda$  if and only if it is a Siegel point for  $f(z) = \lambda z + z^2$ . See also [P9].

**Remark 4.2:** Condition (4.6) is usually expressed in a different way. Write  $\lambda = e^{2\pi i\theta}$ , and let  $\{p_k/q_k\}$  be the sequence of rational numbers converging to  $\theta$  given by the expansion in continued fractions. Then (4.6) is equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty,$$

while (4.5) is equivalent to  $q_{n+1} = O(q_n^\beta)$ , and (4.4) is equivalent to

$$\limsup_{k \rightarrow +\infty} \frac{1}{q_k} \log q_{k+1} = +\infty.$$

See [He], [Y2] and references therein for details.

If 0 is a Siegel point for  $f \in \text{End}(\mathbb{C}, 0)$ , the local dynamics of  $f$  is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of  $f$ , then the local dynamics of  $f$  is very complicated and not yet completely understood. Pérez-Marco (in [P2, 4–7]) has studied the topology and the dynamics of the stable set in this case. Some of his results are summarized in the following

**Theorem 4.6:** (Pérez-Marco, 1995 [P6, 7]) *Assume that 0 is a Cremer point for an elliptic holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}, 0)$ . Then:*

- (i) *The stable set  $K_f$  is compact, connected, full (i.e.,  $\mathbb{C} \setminus K_f$  is connected), it is not reduced to  $\{0\}$ , and it is not locally connected at any point distinct from the origin.*
- (ii) *Any point of  $K_f \setminus \{0\}$  is recurrent (that is, a limit point of its orbit).*
- (iii) *There is an orbit in  $K_f$  which accumulates at the origin, but no non-trivial orbit converges to the origin.*

**Remark 4.3:** As far as I know, there are neither a topological nor a holomorphic complete classification of elliptic holomorphic dynamical systems with a Cremer point. Furthermore, if  $\lambda \in S^1$  is not a root of unity and does not satisfy Bryuno’s condition (4.6), we can find  $f_1, f_2 \in \text{End}(\mathbb{C}, 0)$  with multiplier  $\lambda$  such that  $f_1$  is holomorphically linearizable while  $f_2$  is not. Then  $f_1$  and  $f_2$  are formally conjugated without being neither holomorphically nor topologically locally conjugated.

See also [P1, 3] for other results on the dynamics about a Cremer point.

## 5. Several complex variables: the hyperbolic case

Now we start the discussion of local dynamics in several complex variables. In this case the theory is much less complete than its one-variable counterpart.

Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system at  $O \in \mathbb{C}^n$ , with  $n \geq 2$ . We can write  $f$  using a *homogeneous expansion*

$$f(z) = P_1(z) + P_2(z) + \cdots \in \mathbb{C}_0\{z_1, \dots, z_n\}^n,$$

where  $P_j$  is an  $n$ -uple of homogeneous polynomials of degree  $j$ . In particular,  $P_1$  is the differential  $df_O$  of  $f$  at the origin, and  $f$  is locally invertible if and only if  $P_1$  is invertible.

We have seen that in dimension one the multiplier (i.e., the derivative at the origin) plays a main rôle. When  $n > 1$ , a similar rôle is played by the eigenvalues of the differential. Thus we introduce the following definitions:

- if all eigenvalues of  $df_O$  have modulus less than 1, we say that the fixed point  $O$  is *attracting*;
- if all eigenvalues of  $df_O$  have modulus greater than 1, we say that the fixed point  $O$  is *repelling*;
- if all eigenvalues of  $df_O$  have modulus different from 1, we say that the fixed point  $O$  is *hyperbolic* (notice that we allow the eigenvalue zero);
- if all eigenvalues of  $df_O$  are roots of unity, we say that the fixed point  $O$  is *parabolic*; in particular, if  $df_O = \text{id}$  we say that  $f$  is *tangent to the identity*;
- if all eigenvalues of  $df_O$  have modulus 1 but none is a root of unity, we say that the fixed point  $O$  is *elliptic*;
- if  $df_O = O$ , we say that the fixed point  $O$  is *superattracting*.

Other cases are clearly possible, but for our aims this list is enough. In this survey we shall be mainly concerned with hyperbolic and parabolic fixed points; however, in the last section we shall also present some results valid in other cases.

Let us begin assuming that the origin is a hyperbolic fixed point for an  $f \in \text{End}(\mathbb{C}^n, O)$  not necessarily invertible. We then have a canonical splitting

$$\mathbb{C}^n = E^s \oplus E^u,$$

where  $E^s$  (respectively,  $E^u$ ) is the direct sum of the generalized eigenspaces associated to the eigenvalues of  $df_O$  with modulus less (respectively, greater) than 1. Then the first main result in this subject is the famous *stable manifold theorem* (originally due to Perron [Pe] and Hadamard [H]; see [FHY, HK, HPS, Pes, Sh] for proofs in the  $C^\infty$  category, Wu [Wu] for a proof in the holomorphic category, and [A3] for a proof in the non-invertible case):

**Theorem 5.1:** *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system with a hyperbolic fixed point at the origin. Then:*

- (i) *the stable set  $K_f$  is an embedded complex submanifold of (a neighbourhood of the origin in)  $\mathbb{C}^n$ , tangent to  $E^s$  at the origin;*
- (ii) *there is an embedded complex submanifold  $W_f$  of (a neighbourhood of the origin in)  $\mathbb{C}^n$ , called the unstable set of  $f$ , tangent to  $E^u$  at the origin, such that  $f|_{W_f}$  is invertible,  $f^{-1}(W_f) \subseteq W_f$ , and  $z \in W_f$  if and only if there is a sequence  $\{z_{-k}\}_{k \in \mathbb{N}}$  in the domain of  $f$  such that  $z_0 = z$  and  $f(z_{-k}) = z_{-k+1}$  for all  $k \geq 1$ . Furthermore, if  $f$  is invertible then  $W_f$  is the stable set of  $f^{-1}$ .*

The proof is too involved to be summarized here; it suffices to say that both  $K_f$  and  $W_f$  can be recovered, for instance, as fixed points of a suitable contracting operator in an infinite dimensional space (see the references quoted above for details).

**Remark 5.1:** If the origin is an attracting fixed point, then  $E^s = \mathbb{C}^n$ , and  $K_f$  is an open neighbourhood of the origin, its *basin of attraction*. However, as we shall discuss below, this does not imply that  $f$  is holomorphically linearizable, not even when it is invertible. Conversely, if the origin is a repelling fixed point, then  $E^u = \mathbb{C}^n$ , and  $K_f = \{O\}$ . Again, not all holomorphic local dynamical systems with a repelling fixed point are holomorphically linearizable.

If a point in the domain  $U$  of a holomorphic local dynamical system with a hyperbolic fixed point does not belong either to the stable set or to the unstable set, it escapes both in forward time (that is, its orbit escapes) and in backward time (that is, it is not the end point of an infinite orbit contained in  $U$ ). In some sense, we can think of the stable and unstable sets (or, as they are usually called in this setting, stable and unstable *manifolds*) as skewed coordinate planes at the origin, and the orbits outside these coordinate planes follow some sort of hyperbolic path, entering and leaving any neighbourhood of the origin in finite time.

Actually, this idea of straightening stable and unstable manifolds can be brought to fruition (at least in the invertible case), and it yields one of the possible proofs (see [HK, Sh, A3] and references therein) of the *Grobman-Hartman theorem*:

**Theorem 5.2:** (Grobman, 1959 [G1–2]; Hartman, 1960 [Har]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then  $f$  is topologically locally conjugated to its differential  $df_O$ .*

Thus, at least from a topological point of view, the local dynamics about an invertible hyperbolic fixed point is completely clear. This is definitely not the case if the local dynamical system is not invertible in a neighbourhood of the fixed point. For instance, already Hubbard and Papadopol [HP] noticed that a Böttcher-type theorem for superattracting points in several complex variables is just not true: there are holomorphic local dynamical systems with a superattracting fixed point which are not even topologically locally conjugated to the first non-vanishing term of their homogeneous expansion. Recently, Favre and Jonsson [FJ] have begun a very detailed study of superattracting fixed points in  $\mathbb{C}^2$ , study that should lead to their topological classification.

The holomorphic and even the formal classification are not as simple as the topological one. The main problem is that, if we denote by  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  the eigenvalues of  $df_O$ , then it may happen that

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \quad (5.1)$$

for some  $1 \leq j \leq n$  and some  $k_1, \dots, k_n \in \mathbb{N}$  with  $k_1 + \cdots + k_n \geq 2$ ; a relation of this kind is called a *resonance* of  $f$ . When  $n = 1$  there is a resonance if and only if the multiplier is a root of unity, or zero; but if  $n > 1$  resonances may occur in the hyperbolic case too. Anyway, a computation completely analogous to the one yielding Proposition 4.2 proves the following

**Proposition 5.3:** *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a (locally invertible) holomorphic local dynamical system with a hyperbolic fixed point and no resonances. Then  $f$  is formally conjugated to its differential  $df_O$ .*

In presence of resonances, even the formal classification is not that easy. Let us assume, for simplicity, that  $df_O$  is in Jordan form, that is

$$P_1(z) = (\lambda_1 z, \epsilon_2 z_1 + \lambda_2 z_2, \dots, \epsilon_n z_{n-1} + \lambda_n z_n),$$

with  $\epsilon_1, \dots, \epsilon_{n-1} \in \{0, 1\}$ . We shall say that a monomial  $z_1^{k_1} \cdots z_n^{k_n}$  in the  $j$ -th coordinate of  $f$  is *resonant* if  $k_1 + \cdots + k_n \geq 2$  and  $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_j$ . Then the Proposition 5.3 can be generalized to

**Proposition 5.4:** *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then it is formally conjugated to a  $g \in \mathbb{C}_0[[z_1, \dots, z_n]]^n$  such that  $dg_O$  is in Jordan normal form, and  $g$  has only resonant monomials.*

The formal series  $g$  is called *Poincaré-Dulac normal form* of  $f$ ; see Arnold [Ar] for a proof of Proposition 5.4.

The problem with Poincaré-Dulac normal forms is that they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including *finitely many* resonant monomials only (as happened, for instance, in Proposition 3.3). This is indeed the case (see, e.g., Reich [R1]) when  $df_O$  belongs to the so-called *Poincaré domain*, that is when  $df_O$  is invertible and  $O$  is either attracting or repelling (when  $df_O$  is still invertible but does not belong to the Poincaré domain, we shall say that it belongs to the *Siegel domain*). As far as I know, the problem of finding canonical formal normal forms when  $df_O$  belongs to the Siegel domain (and  $f$  is hyperbolic) is still open.

It should be remarked that, in the hyperbolic case, the problem of formal linearization is equivalent to the problem of smooth linearization. This has been proved by Sternberg [St1–2] and Chaperon [Ch]:

**Theorem 5.5:** (Sternberg, 1957 [St1–2]; Chaperon, 1986 [Ch]) *Let  $f, g \in \text{End}(\mathbb{C}^n, O)$  be two holomorphic local dynamical systems, and assume that  $f$  is locally invertible and with a hyperbolic fixed point at the origin. Then  $f$  and  $g$  are formally conjugated if and only if they are smoothly locally conjugated. In particular,  $f$  is smoothly linearizable if and only if it is formally linearizable. Thus if there are no resonances then  $f$  is smoothly linearizable.*

Even without resonances, the holomorphic linearizability is not guaranteed. The easiest positive result is due to Poincaré [Po] who, using majorant series, proved the following

**Theorem 5.6:** (Poincaré, 1893 [Po]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a locally invertible holomorphic local dynamical system with an attracting or repelling fixed point. Then  $f$  is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then  $f$  is holomorphically linearizable.*

Reich [R2] describes holomorphic normal forms when  $df_O$  belongs to the Poincaré domain and there are resonances (see also [ÉV]); Pérez-Marco [P8] discusses the problem of holomorphic linearization in the presence of resonances.

When  $df_O$  belongs to the Siegel domain, even without resonances, the formal linearization might diverge. To describe the known results, let us introduce the following quantity:

$$\Omega_{\lambda_1, \dots, \lambda_n}(m) = \min\{|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j| \mid k_1, \dots, k_n \in \mathbb{N}, 2 \leq k_1 + \cdots + k_n \leq m, 1 \leq j \leq n\}$$

for  $m \geq 2$  and  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ . In particular, if  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $df_O$ , we shall write  $\Omega_f(m)$  for  $\Omega_{\lambda_1, \dots, \lambda_n}(m)$ .

It is clear that  $\Omega_f(m) \neq 0$  for all  $m \geq 2$  if and only if there are no resonances. It is also not difficult to prove that if  $df_O$  belongs to the Siegel domain then

$$\lim_{m \rightarrow +\infty} \Omega_f(m) = 0,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case. As far as I know, the best positive and negative results in this setting are due to Bryuno [Bry2–3], and are a natural generalization of their one-dimensional counterparts:

**Theorem 5.7:** (Bryuno, 1971 [Bry2–3]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system such that  $df_O$  belongs to the Siegel domain, is linearizable and has no resonances. Assume moreover that*

$$\sum_{k=0}^{+\infty} \left( -\frac{1}{2^k} \log \Omega_f(2^{k+1}) \right) < +\infty. \quad (5.2)$$

*Then  $f$  is holomorphically linearizable.*

**Theorem 5.8:** *Let  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  be without resonances and such that*

$$\limsup_{m \rightarrow +\infty} \left( -\frac{1}{m} \log \Omega_{\lambda_1, \dots, \lambda_n}(m) \right) = +\infty.$$

*Then there exists  $f \in \text{End}(\mathbb{C}^n, O)$ , with  $df_O = \text{diag}(\lambda_1, \dots, \lambda_n)$ , not holomorphically linearizable.*

**Remark 5.2:** These theorems hold even without hyperbolicity assumptions.

It should be remarked that, contrarily to the one-dimensional case, it is not known whether condition (5.2) is necessary for the holomorphic linearizability of all holomorphic local dynamical systems with a given linear part belonging to the Siegel domain. See also Pöschel [Pö] for a generalization of Theorem 5.7, and Il'yachenko [I1] for an important result related to Theorem 5.8. Finally, in [DG] are discussed results in the spirit of Theorem 5.7 without assuming that the differential is diagonalizable.

## 6. Several complex variables: the parabolic case

A first natural question in the several complex variables parabolic case is whether a result like the Leau-Fatou flower theorem holds, and, if so, in which form. To present what is known on this subject in this section we shall restrict our attention to holomorphic local dynamical systems tangent to the identity; consequences on dynamical systems with a more general parabolic fixed point can be deduced taking a suitable iterate (but see also the end of this section for results valid when the differential at the fixed point is not diagonalizable).

So we are interested in the local dynamics of a holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}^n, O)$  of the form

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots \in \mathbb{C}_0\{z_1, \dots, z_n\}^n,$$

where  $P_\nu$  is the first non-zero term in the homogeneous expansion of  $f$ ; the number  $\nu \geq 2$  is the *order* of  $f$ .

The two main ingredients in the statement of the Leau-Fatou flower theorem were the attracting directions and the petals. Let us first describe a several variables analogue of attracting directions.

Let  $f \in \text{End}(\mathbb{C}^n, O)$  be tangent at the identity and of order  $\nu$ . A *characteristic direction* for  $f$  is a non-zero vector  $v \in \mathbb{C}^n \setminus \{O\}$  such that  $P_\nu(v) = \lambda v$  for some  $\lambda \in \mathbb{C}$ . If  $P_\nu(v) = O$  (that is,  $\lambda = 0$ ) we shall say that  $v$  is a *degenerate* characteristic direction; otherwise, (that is, if  $\lambda \neq 0$ ) we shall say that  $v$  is *non-degenerate*.

There is an equivalent definition of characteristic directions that shall be useful later on. The  $n$ -uple of  $\nu$ -homogeneous polynomial  $P_\nu$  induces a meromorphic self-map of  $\mathbb{P}^{n-1}(\mathbb{C})$ , still denoted by  $P_\nu$ . Then, under the canonical projection  $\mathbb{C}^n \setminus \{O\} \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$  that we shall denote by  $v \mapsto [v]$ , the non-degenerate characteristic directions correspond exactly to fixed points of  $P_\nu$ , and the degenerate characteristic directions correspond exactly to indeterminacy points of  $P_\nu$ . By the way, using Bezout's theorem it is easy to prove (see, e.g., [AT]) that the number of characteristic directions of  $f$ , counted according to a suitable multiplicity, is given by  $(\nu^n - 1)/(\nu - 1)$ .

**Remark 6.1:** The characteristic directions are *complex* directions; in particular, it is easy to check that  $f$  and  $f^{-1}$  have the same characteristic directions. Later we shall see how to associate to (most) characteristic directions  $\nu - 1$  petals, each one in some sense centered about a *real* attracting direction corresponding to the same complex characteristic direction.

The notion of characteristic directions has a dynamical origin. We shall say that an orbit  $\{f^k(z_0)\}$  converges to the origin *tangentially* to a direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$  if  $f^k(z_0) \rightarrow O$  in  $\mathbb{C}^n$  and  $[f^k(z_0)] \rightarrow [v]$  in  $\mathbb{P}^{n-1}(\mathbb{C})$ . Then

**Proposition 6.1:** *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic dynamical system tangent to the identity. If there is an orbit of  $f$  converging to the origin tangentially to a direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ , then  $v$  is a characteristic direction of  $f$ .*

*Sketch of proof:* ([Ha2]) For simplicity let us assume  $\nu = 2$ ; a similar argument works for  $\nu > 2$ .

If  $v$  is a degenerate characteristic direction, there is nothing to prove. If not, up to a linear change of coordinates we can write

$$\begin{cases} f_1(z) = z_1 + p_2^1(z_1, z') + p_3^1(z_1, z') + \cdots, \\ f'(z) = z' + p_2'(z_1, z') + p_3'(z_1, z') + \cdots, \end{cases}$$

where  $z' = (z_2, \dots, z_n) \in \mathbb{C}^{n-1}$ ,  $f = (f_1, f')$ ,  $P_j = (p_j^1, p_j')$  and so on, with  $v = (1, v')$  and  $p_2^1(1, v') \neq 0$ . Making the substitution

$$\begin{cases} w_1 = z_1, \\ z' = w' z_1, \end{cases} \quad (6.1)$$

which is a change of variable outside the hyperplane  $z_1 = 0$ , the map  $f$  becomes

$$\begin{cases} \tilde{f}_1(w) = w_1 + p_2^1(1, w')w_1^2 + p_3^1(1, w')w_1^3 + \cdots, \\ \tilde{f}'(w) = w' + r(w')w_1 + O(w_1^2), \end{cases} \quad (6.2)$$

where  $r(w')$  is a polynomial such that  $r(v') = O$  if and only if  $(1, v')$  is a characteristic direction of  $f$  with  $p_2^1(1, v') \neq 0$ .

Now, the hypothesis is that there exists an orbit  $\{f^k(z_0)\}$  converging to the origin and such that  $[f^k(z_0)] \rightarrow [v]$ . Writing  $\tilde{f}^k(w_0) = (w_1^k, (w')^k)$ , this implies that  $w_1^k \rightarrow 0$  and  $(w')^k \rightarrow v'$ . Then it is not difficult to prove that

$$\lim_{k \rightarrow +\infty} \frac{1}{kw_1^k} = -p_2^1(1, v')$$

and then that  $(w')^{k+1} - (w')^k$  is of the order of  $r(v')/k$ , which implies  $r(v') = O$ , as claimed.  $\square$

**Remark 6.2:** There are (unfortunately?) examples of  $f \in \text{End}(\mathbb{C}^2, O)$  tangent to the identity with an orbit converging to the origin which is not tangent to any direction (see [Ri1]).

The several variables analogue of a petal is instead given by the notion of parabolic curve. A *parabolic curve* for  $f \in \text{End}(\mathbb{C}^n, O)$  tangent to the identity is an injective holomorphic map  $\varphi: \Delta \rightarrow \mathbb{C}^n \setminus \{O\}$  satisfying the following properties:

- (a)  $\Delta$  is a simply connected domain in  $\mathbb{C}$  with  $0 \in \partial\Delta$ ;
- (b)  $\varphi$  is continuous at the origin, and  $\varphi(0) = O$ ;
- (c)  $\varphi(\Delta)$  is  $f$ -invariant, and  $(f|_{\varphi(\Delta)})^k \rightarrow O$  uniformly on compact subsets as  $k \rightarrow +\infty$ .

Furthermore, if  $[\varphi(\zeta)] \rightarrow [v]$  in  $\mathbb{P}^{n-1}(\mathbb{C})$  as  $\zeta \rightarrow 0$  in  $\Delta$ , we shall say that the parabolic curve  $\varphi$  is *tangent* to the direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ .

Then the first main generalization of the Leau-Fatou flower theorem to several complex variables is

**Theorem 6.2:** (Écalle, 1985 [É3]; Hakim, 1998 [Ha2]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system tangent to the identity of order  $\nu \geq 2$ . Then for any non-degenerate characteristic direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$  there exist (at least)  $\nu - 1$  parabolic curves for  $f$  tangent to  $[v]$ .*

*Sketch of proof:* Écalle proof is based on his theory of resurgence of divergent series; we shall describe here the ideas behind Hakim's proof, which depends on more standard arguments.

For the sake of simplicity, let us assume  $n = 2$ ; without loss of generality we can also assume  $[v] = [1 : 0]$ . Then after a linear change of variables and a transformation of the kind (6.1) we are reduced to prove the existence of a parabolic curve at the origin for a map of the form

$$\begin{cases} f_1(z) = z_1 - z_1^\nu + O(z_1^{\nu+1}, z_1^\nu z_2), \\ f_2(z) = z_2(1 - \lambda z_1^{\nu-1} + O(z_1^\nu, z_1^{\nu-1} z_2)) + z_1^\nu \psi(z), \end{cases}$$

where  $\psi$  is holomorphic with  $\psi(O) = 0$ , and  $\lambda \in \mathbb{C}$ . Given  $\delta > 0$ , set  $D_{\delta, \nu} = \{\zeta \in \mathbb{C} \mid |\zeta^{\nu-1} - \delta| < \delta\}$ ; this open set has  $\nu - 1$  connected components, all of them satisfying condition (a) on the domain of a parabolic curve. Furthermore, if  $u$  is a holomorphic function defined on one of these connected components, of the form  $u(\zeta) = \zeta^2 u_o(\zeta)$  for some bounded holomorphic function  $u_o$ , and such that

$$u(f_1(\zeta, u(\zeta))) = f_2(\zeta, u(\zeta)), \quad (6.3)$$

then it is not difficult to verify that  $\varphi(\zeta) = (\zeta, u(\zeta))$  is a parabolic curve for  $f$  tangent to  $[v]$ .

So we are reduced to finding a solution of (6.3) in each connected component of  $D_{\delta, \nu}$ , with  $\delta$  small enough. For any holomorphic  $u = \zeta^2 u_o$  defined in such a connected component, let  $f_u(\zeta) = f_1(\zeta, u(\zeta))$ , put

$$H(z) = z_2 - \frac{z_1^\lambda}{f_1(z)^\lambda} f_2(z),$$

and define the operator  $T$  by setting

$$(Tu)(\zeta) = \zeta^\lambda \sum_{k=0}^{\infty} \frac{H(f_u^k(\zeta), u(f_u^k(\zeta)))}{f_u^k(\zeta)^\lambda}.$$

Then, if  $\delta > 0$  is small enough, it is possible to prove that  $T$  is well-defined, that  $u$  is a fixed point of  $T$  if and only if it satisfies (6.3), and that  $T$  is a contraction of a closed convex set of a suitable complex Banach space — and thus it has a fixed point. In this way if  $\delta > 0$  is small enough we get a unique solution of (6.3) for each connected component of  $D_{\delta, \nu}$ , and hence  $\nu - 1$  parabolic curves tangent to  $[v]$ .  $\square$

A set of  $\nu - 1$  parabolic curves obtained in this way will be called a *Fatou flower* for  $f$  tangent to  $[v]$ .

**Remark 6.3:** It should be remarked that a similar result for 2-dimensional maps with  $\lambda \notin \mathbb{N}^*$  has been obtained by Weickert [W] too; the computations needed in the proof for the case  $\lambda \in \mathbb{N}^*$  are considerably harder, and were not carried out by him.

**Remark 6.4:** When there is a one-dimensional  $f$ -invariant complex submanifold passing through the origin tangent to a characteristic direction  $[v]$ , the previous theorem is just a consequence of the usual one-dimensional theory. But it turns out that in most cases such an  $f$ -invariant complex submanifold does not exist: see [Ha2] for a concrete example, and [É3] for a general discussion.

We can also have  $f$ -invariant complex submanifolds of dimension strictly greater than one still attracted by the origin. Given a holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}^n, O)$  tangent to the identity of order  $\nu \geq 2$ , and a non-degenerate characteristic direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ , the eigenvalues  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$  of the linear operator  $d(P_\nu)_{[v]} - \text{id}: T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \rightarrow T_{[v]}\mathbb{P}^{n-1}(\mathbb{C})$  will be called the *directors* of  $[v]$ . Then, using a more elaborate version of her proof of Theorem 6.2, Hakim has been able to prove the following:

**Theorem 6.3:** (Hakim, 1997 [Ha3]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system tangent to the identity of order  $\nu \geq 2$ . Let  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$  be a non-degenerate characteristic direction, with directors  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$ . Furthermore, assume that  $\text{Re } \alpha_1, \dots, \text{Re } \alpha_d > 0$  and  $\text{Re } \alpha_{d+1}, \dots, \text{Re } \alpha_{n-1} \leq 0$  for a suitable  $d \geq 0$ . Then:*

- (i) *There exists an  $f$ -invariant  $(d+1)$ -dimensional complex submanifold  $M$  of  $\mathbb{C}^n$ , with the origin in its boundary, such that the orbit of every point of  $M$  converges to the origin tangentially to  $[v]$ ;*
- (ii)  *$f|_M$  is holomorphically conjugated to the translation  $\tau(w_0, w_1, \dots, w_d) = (w_0 + 1, w_1, \dots, w_d)$  defined on a suitable right half-space in  $\mathbb{C}^{d+1}$ .*

**Remark 6.5:** In particular, if all the directors of  $[v]$  have positive real part, there is an open domain attracted by the origin. However, the condition given by Theorem 6.3 is not necessary for the existence of such an open domain; see Rivi [Ri1] for an easy example, and Ushiki [Us] for a more elaborate example with an open domain attracted by the origin where  $f$  cannot be conjugate to a translation.

In his monumental work [É3] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with at least one non-degenerate characteristic direction. For instance, he has proved the following

**Theorem 6.4:** (Écalle, 1985 [É3]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system tangent to the identity of order  $\nu \geq 2$ . Assume that*

- (a)  *$f$  has exactly  $(\nu^n - 1)/(\nu - 1)$  distinct non-degenerate characteristic directions and no degenerate characteristic directions;*
- (b) *the directors of any non-degenerate characteristic direction are irrational and mutually independent over  $\mathbb{Z}$ .*

*Choose a non-degenerate characteristic direction  $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ , and let  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$  be its directors. Then there exist a unique  $\rho \in \mathbb{C}$  and unique (up to dilations) formal series  $R_1, \dots, R_n \in \mathbb{C}[[z_1, \dots, z_n]]$ , where each  $R_j$  contains only monomial of total degree at least  $\nu + 1$  and of partial degree in  $z_j$  at most  $\nu - 2$ , such that  $f$  is formally conjugated to the time-1 map of the formal vector field*

$$X = \frac{1}{(\nu - 1)(1 + \rho z_n^{\nu-1})} \left\{ [-z_n^\nu + R_n(z)] \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} [-\alpha_j z_n^{\nu-1} z_j + R_j(z)] \frac{\partial}{\partial z_j} \right\}.$$

Another approach to the formal classification, at least in dimension 2, is described in [BM].

Furthermore, using his theory of resurgence, and always assuming the existence of at least one non-degenerate characteristic direction, Écalle has also provided a set of holomorphic invariants for holomorphic local dynamical systems tangent to the identity, in terms of differential operators with formal power series as coefficients. Moreover, if the directors of all non-degenerate characteristic direction are irrational and satisfy a suitable diophantine condition, then these invariants become a complete set of invariants. See [É4] for a description of his results, and [É3] for the details.

Now, all these results beg the question: what happens when there are no non-degenerate characteristic directions? For instance, this is the case for

$$\begin{cases} f_1(z) = z_1 + bz_1z_2 + z_2^2, \\ f_2(z) = z_2 - b^2z_1z_2 - bz_2^2 + z_1^3, \end{cases}$$

for any  $b \in \mathbb{C}^*$ , and it is easy to build similar examples of any order. At present, the theory in this case is satisfactorily developed for  $n = 2$  only. In particular, in [A2] is proved the following

**Theorem 6.5:** (Abate, 2001 [A2]) *Every holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}^2, O)$  tangent to the identity, with an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

**Remark 6.6:** Bracci and Suwa have proved a version of Theorem 6.5 for  $f \in \text{End}(M, p)$  where  $M$  is a singular variety with not too bad a singularity at  $p$ ; see [BrS] for details.

Let us describe the main ideas in the proof of Theorem 6.5, because they provide some insight on the dynamical structure of holomorphic local dynamical systems tangent to the identity, and on how to deal with it.

The first idea is to exploit in a systematic way the transformation (6.1), following a procedure borrowed from algebraic geometry. If  $p$  is a point in a complex manifold  $M$ , there is a canonical way to build a complex manifold  $\tilde{M}$ , called the *blow-up* of  $M$  at  $p$ , provided with a holomorphic projection  $\pi: \tilde{M} \rightarrow M$ , and such that  $E = \pi^{-1}(p)$ , the *exceptional divisor* of the blow-up, is canonically biholomorphic to  $\mathbb{P}(T_p M)$ , and  $\pi|_{\tilde{M} \setminus E}: \tilde{M} \setminus E \rightarrow M \setminus \{p\}$  is a biholomorphism. In suitable local coordinates, the map  $\pi$  is exactly given by (6.1). Furthermore, if  $f \in \text{End}(M, p)$  is tangent to the identity, there is a unique way to lift  $f$  to a map  $\tilde{f} \in \text{End}(\tilde{M}, E)$  such that  $\pi \circ \tilde{f} = f \circ \pi$ , where  $\text{End}(\tilde{M}, E)$  is the set of holomorphic maps defined in a neighbourhood of  $E$  with values in  $\tilde{M}$  and which are the identity on  $E$ . In particular, the characteristic directions of  $f$  become points in the domain of  $\tilde{f}$ .

This approach allows to determine which characteristic directions are dynamically meaningful. Take  $f = (f_1, f_2) \in \text{End}(\mathbb{C}^2, O)$  tangent to the identity; if  $\ell = \gcd(f_1 - z_1, f_2 - z_2)$ , we can write

$$f_j(z) = z_j + \ell(z)g_j(z)$$

for  $j = 1, 2$ , with  $g_1$  and  $g_2$  relatively prime in  $\mathbb{C}\{z_1, z_2\}$ . We shall say that  $O$  is a *singular point* for  $f$  if  $g_1(O) = g_2(O) = 0$ . Clearly, if  $O$  is an isolated fixed point of  $f$  then it is singular; but if  $O$  is not an isolated fixed point (i.e.,  $\ell \neq 1$ ) it might not be singular. Only singular points are dynamically meaningful, because a not too difficult computation (see [A2], and [AT] for an  $n$ -dimensional generalization) yields the following

**Proposition 6.6:** *Let  $f \in \text{End}(\mathbb{C}^2, O)$  be a holomorphic local dynamical system tangent to the identity. If the fixed point  $O$  is not singular, then  $K_f$  reduces to the fixed point set of  $f$ .*

Now, if  $\tilde{M}$  is the blow-up of  $\mathbb{C}^2$  at the origin, then the lift  $\tilde{f}$  of  $f$  belongs to  $\text{End}(\tilde{M}, [v])$  for any direction  $[v] \in \mathbb{P}^1(\mathbb{C}) = E$ . We shall then say that  $[v] \in \mathbb{P}^1(\mathbb{C})$  is a *singular direction* of  $f$  if it is a singular point for  $\tilde{f}$ . It turns out that non-degenerate characteristic directions are always singular (but the converse does not necessarily hold), and that singular directions are always characteristic (but the converse does not necessarily hold): the singular directions are the dynamically interesting characteristic directions.

The important feature of the blow-up procedure is that even though the underlying manifold becomes more complex, the lifted maps become simpler. Indeed, using an argument similar to one (described, for instance, in [MM]) used in the study of singular holomorphic foliations of 2-dimensional complex manifolds, in [A2] it is shown that after a finite number of blow-ups our original holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}^2, O)$  can be lifted to a map  $\tilde{f}$  whose singular points (are finitely many and) satisfy one of the following conditions:

- (o) they are *dicritical*, that is with infinitely many singular directions; or,
- ( $\star$ ) in suitable local coordinates centered at the singular point we can write

$$\begin{cases} \tilde{f}_1(z) = z_1 + \ell(z)(\lambda_1 z_1 + O(\|z\|^2)), \\ \tilde{f}_2(z) = z_2 + \ell(z)(\lambda_2 z_2 + O(\|z\|^2)), \end{cases}$$

with

- ( $\star_1$ )  $\lambda_1, \lambda_2 \neq 0$  and  $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \mathbb{N}$ , or
- ( $\star_2$ )  $\lambda_1 \neq 0, \lambda_2 = 0$ .

**Remark 6.7:** This “reduction of the singularities” statement holds only in dimension 2, and it is not clear how to replace it in higher dimensions.

It is not too difficult to prove that Theorem 6.2 (actually, the “easy” case of this theorem) can be applied both to dicritical and to ( $\star_1$ ) singularities; therefore as soon as this blow-up procedure produces such a singularity, we get a Fatou flower for the original dynamical system  $f$ .

So to end the proof of Theorem 6.5 it suffices to prove that any such blow-up procedure *must* produce at least one dicritical or ( $\star_1$ ) singularity. To get such a result, we need a completely new ingredient.

Let  $E$  be a compact Riemann surface inside a 2-dimensional complex manifold (for instance,  $E$  can be the exceptional divisor of the blow-up of a point  $p$ ), and take  $f \in \text{End}(M, E)$  tangent to the identity to all points of  $E$  (this happens, for instance, if  $f$  is the lifting of a map tangent at the identity at  $p$ ). Given  $q \in E$ ,



choose local coordinates  $(z_1, z_2)$  in  $M$  centered at  $q$  and such that  $E$  is locally given by  $\{z_2 = 0\}$ . Then the function

$$k(z_1) = \lim_{z_2 \rightarrow 0} \frac{f_2(z) - z_2}{z_2(f_1(z) - z_1)}$$

is either a meromorphic function defined in a neighbourhood of  $q$ , or identically  $\infty$ . It turns out that:

- if  $k$  is identically  $\infty$  at one point  $q \in E$ , it is identically  $\infty$  at all points of  $E$ ; in this case we shall say that  $f$  is *not tangential* to  $E$ ;
- if  $f$  is tangential to  $E$  (this happens, for instance, if  $f$  is obtained blowing up a non-dicritical singularity), then the residue of  $k$  at  $q$  is independent of the local coordinates used to define  $k$ , and it is called the *index*  $\iota_q(f, E)$  of  $f$  at  $q$  along  $E$ ;
- if  $f$  is tangential to  $E$ , and  $q \in E$  is not singular for  $f$ , then  $\iota_q(f, E) = 0$ ; in particular,  $\iota_q(f, E) \neq 0$  only for a finite number of points of  $E$ .

Then following an argument suggested by Camacho and Sad [CS] in their study of the separatrices of holomorphic foliations it is possible to prove the following *index theorem*:

**Theorem 6.7:** (Abate, 2001 [A2]) *Let  $E$  be a compact Riemann surface inside a 2-dimensional complex manifold  $M$ . Take  $f \in \text{End}(M, E)$  such that  $f$  is tangent to the identity at all points of  $E$ , and assume that  $f$  is tangential to  $E$ . Then*

$$\sum_{q \in E} \iota_q(f, E) = c_1(N_E),$$

where  $c_1(N_E)$  is the first Chern class of the normal bundle  $N_E$  of  $E$  in  $M$ .

**Remark 6.8:** If  $f$  is the lift to the blow-up of a map tangent to the identity, and  $[v] \in E$  is a non-degenerate characteristic direction with non-zero director  $\alpha$ , then  $\iota_{[v]}(f, E) = 1/\alpha$ .

**Remark 6.9:** Theorem 6.7 is only a very particular case of a much more general index theorem, valid for holomorphic self-maps of complex manifolds of any dimension fixing pointwise a smooth complex submanifold of any codimension, or a hypersurface even with singularities; see [BrT], [Br] and [ABT], where some applications to dynamics are also discussed. In particular, in [ABT] is introduced a canonical section of a suitable vector bundle describing the local dynamics in an infinitesimal neighbourhood of the submanifold, providing in particular a more intrinsic description of the index.

Now, a combinatorial argument (again inspired by Camacho and Sad [CS]) shows that if we have  $f \in \text{End}(\mathbb{C}^2, O)$  with an isolated fixed point, and such that applying the blow-up procedure to the lifted map  $\tilde{f}$  starting from a singular direction  $[v] \in \mathbb{P}^1(\mathbb{C}) = E$  we end up with only  $(\star_2)$  singularities, then the index of  $\tilde{f}$  at  $[v]$  along  $E$  must be a non-negative rational number. But the first Chern class of  $N_E$  is  $-1$ , and so there must be at least one singular directions whose index is not a non-negative rational number, and thus the blow-up procedure must yield at least one dicritical or  $(\star_1)$  singularity, and hence a Fatou flower for our map  $f$ , completing the proof of Theorem 6.5.

Actually, we have proved the following slightly more precise result:

**Theorem 6.8:** (Abate, 2001 [A2]) *Let  $f \in \text{End}(\mathbb{C}^2, O)$  be a holomorphic local dynamical system tangent to the identity and with an isolated fixed point at the origin. Let  $[v] \in \mathbb{P}^1(\mathbb{C})$  be a singular direction such that  $\iota_{[v]}(\tilde{f}, \mathbb{P}^1(\mathbb{C})) \notin \mathbb{Q}^+$ , where  $\tilde{f}$  is the lift of  $f$  to the blow-up of the origin. Then  $f$  has a Fatou flower tangent to  $[v]$ .*

**Remark 6.10:** To be even more precise, Theorem 6.8 is more a statement on the lifted map  $\tilde{f}$  than on the original  $f$ . Indeed, the argument used to prove Theorem 6.8 (or a similar argument along the lines of [Ca]) can be used to prove the following: *let  $E$  be a (not necessarily compact) Riemann surface inside a 2-dimensional complex manifold  $M$ , and take  $f \in \text{End}(M, E)$  tangent to the identity at all points of  $E$  and tangential to  $E$ . Let  $p \in E$  be a singular point of  $f$  such that  $\iota_p(f, E) \notin \mathbb{Q}^+$ . Then there exist parabolic curves for  $f$  at  $p$ .* This latter statement has been recently generalized in two ways. Degli Innocenti [DI] has proved that we can allow  $E$  to be singular at  $p$  (but irreducible; in the reducible case one has to impose conditions on the indices of  $f$  along all irreducible components of  $E$  passing through  $p$ ). Molino [Mo], on the

other hand, has proved that the statement still holds assuming only  $\iota_p(f, E) \neq 0$ , at least for  $f$  of order 2 (and  $E$  smooth at  $p$ ); it is natural to conjecture that this should be true for  $f$  of any order.

As already remarked, the reduction of singularities via blow-ups seem to work only in dimension 2. This leaves open the problem of the validity of something like Theorem 6.5 in dimension  $n \geq 3$ ; see [AT] for some partial results.

Furthermore, as far as I know, it is completely open, even in dimension 2, the problem of describing the stable set of a holomorphic local dynamical system tangent to the identity, as well as the more general problem of the topological classification of such dynamical systems. Some results in the case of a dicritical singularity are presented in [BM].

We end this section with a couple of words on holomorphic local dynamical systems with a parabolic fixed point where the differential is not diagonalizable. Particular examples are studied in detail in [CD], [A4] and [GS]. In [A1] it is described a canonical procedure for lifting an  $f \in \text{End}(\mathbb{C}^n, O)$  whose differential at the origin is not diagonalizable to a map defined in a suitable iterated blow-up of the origin (obtained blowing-up not only points but more general submanifolds) with a canonical fixed point where the differential is diagonalizable. Using this procedure it is for instance possible to prove the following

**Corollary 6.9:** (Abate, 2001 [A2]) *Let  $f \in \text{End}(\mathbb{C}^2, O)$  be a holomorphic local dynamical system with  $df_O = J_2$ , the canonical Jordan matrix associated to the eigenvalue 1, and assume that the origin is an isolated fixed point. Then  $f$  admits at least one parabolic curve tangent to  $[1 : 0]$  at the origin.*

## 7. Several complex variables: other cases

Outside the hyperbolic and parabolic cases, there are not that many general results. Theorems 5.7 and 5.8 apply to the elliptic case too, but, as already remarked, it is not known whether the Bryuno condition is still necessary for holomorphic linearizability, that is, if any analogue of Theorem 4.5.(ii) holds in several variables. However, another result in the spirit of Theorem 5.8 is the following:

**Theorem 7.1:** (Yoccoz, 1995 [Y2]) *Let  $A \in GL(n, \mathbb{C})$  be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists  $f \in \text{End}(\mathbb{C}^n, O)$  with  $df_O = A$  which is not holomorphically linearizable.*

A case that has received some attention is the so-called semi-attractive case: a holomorphic local dynamical system  $f \in \text{End}(\mathbb{C}^n, O)$  is said *semi-attractive* if the eigenvalues of  $df_O$  are either equal to 1 or with modulus strictly less than 1. The dynamics of semi-attractive dynamical systems has been studied in detail by Fatou [F4], Nishimura [N], Ueda [U1–2], Hakim [H1] and Rivi [Ri–2]. Their results more or less say that the eigenvalue 1 yields the existence of a “parabolic manifold”  $M$  — in the sense of Theorem 6.3.(ii) — of a suitable dimension, while the eigenvalues with modulus less than one ensure, roughly speaking, that the orbits of points in the normal bundle of  $M$  close enough to  $M$  are attracted to it. For instance, Rivi proved the following

**Theorem 7.2:** (Rivi, 1999 [Ri1–2]) *Let  $f \in \text{End}(\mathbb{C}^n, O)$  be a holomorphic local dynamical system. Assume that 1 is an eigenvalue of (algebraic and geometric) multiplicity  $q \geq 1$  of  $df_O$ , and that the other eigenvalues of  $df_O$  have modulus less than 1. Then:*

- (i) *We can choose local coordinates  $(z, w) \in \mathbb{C}^q \times \mathbb{C}^{n-q}$  such that  $f$  expressed in these coordinates becomes*

$$\begin{cases} f_1(z, w) = A(w)z + P_{2,w}(z) + P_{3,w}(z) + \cdots, \\ f_2(z, w) = G(w) + B(z, w)z, \end{cases}$$

where:  $A(w)$  is a  $q \times q$  matrix with entries holomorphic in  $w$  and  $A(O) = I_q$ ; the  $P_{j,w}$  are  $q$ -uples of homogeneous polynomials in  $z$  of degree  $j$  whose coefficients are holomorphic in  $w$ ;  $G$  is a holomorphic self-map of  $\mathbb{C}^{n-q}$  such that  $G(O) = O$  and the eigenvalues of  $dG_O$  are the eigenvalues of  $df_O$  with modulus strictly less than 1; and  $B(z, w)$  is an  $(n-q) \times q$  matrix of holomorphic functions vanishing at the origin. In particular,  $f_1(z, O)$  is tangent to the identity.

- (ii) *If  $v \in \mathbb{C}^q \subset \mathbb{C}^n$  is a non-degenerate characteristic direction for  $f_1(z, O)$ , and the latter map has order  $\nu$ , then there exist  $\nu - 1$  disjoint  $f$ -invariant  $(n - q + 1)$ -dimensional complex submanifolds  $M_j$  of  $\mathbb{C}^n$ , with the origin in their boundary, such that the orbit of every point of  $M_j$  converges to the origin tangentially*

to  $\mathbb{C}v \oplus E$ , where  $E \subset \mathbb{C}^n$  is the subspace generated by the generalized eigenspaces associated to the eigenvalues of  $df_O$  with modulus less than one.

Rivi also has results in the spirit of Theorem 6.3, and results when the algebraic and geometric multiplicities of the eigenvalue 1 differ; see [Ri1, 2] for details.

As far as I know, there are no results on the formal or holomorphic classification of semi-attractive holomorphic local dynamical systems. However, Canille Martins has given the topological classification in dimension 2, using Theorem 3.2 and general results on normally hyperbolic dynamical systems due to Palis and Takens [PT]:

**Theorem 7.3:** (Canille Martins, 1992 [CM]) *Let  $f \in \text{End}(\mathbb{C}^2, O)$  be a holomorphic local dynamical system such that  $df_O$  has two eigenvalues  $\lambda_1, \lambda_2 \in \mathbb{C}$ , where  $\lambda_1$  is a primitive  $q$ -root of unity, and  $|\lambda_2| \neq 0, 1$ . Then  $f$  is topologically locally conjugated to the map*

$$(z, w) \mapsto (\lambda_1 z + z^{kq+1}, \lambda_2 w)$$

for a suitable  $k \in \mathbb{N}^*$ .

We end this survey by recalling a very recent result by Bracci and Molino. Assume that  $f \in \text{End}(\mathbb{C}^2, O)$  is a holomorphic local dynamical system such that the eigenvalues of  $df_O$  are 1 and  $e^{2\pi i\theta} \neq 1$ . If  $e^{2\pi i\theta}$  satisfies the Bryuno condition, Pöschel [Pö] proved the existence of a 1-dimensional  $f$ -invariant holomorphic disk containing the origin where  $f$  is conjugated to the irrational rotation of angle  $\theta$ . On the other hand, Bracci and Molino give sufficient conditions (depending on  $f$  but not on  $e^{2\pi i\theta}$ , expressed in terms of two new holomorphic invariants, and satisfied by generic maps) for the existence of parabolic curves tangent to the eigenspace of the eigenvalue 1; see [BrM] for details.

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